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Introducing a metric on the space of fuzzy continuous mappings and the completeness of the space

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We consider the space of the mappings which take their values in the set of fuzzy numbers, and introduce a metric on the space. We prove that the space constitutes a complete space under the metric.

A fuzzy number we treat in this paper is as follows.

Definition 1. A *fuzzy number* is a fuzzy set with a membership function $\mu : \mathbf{R} \to [0,1]$ satisfying the following conditions:

(i) there are real numbers a and b such that

$$\mathrm{cl}\{t \in \mathbf{R} \mid \mu(\mathrm{t}) > 0\} = [a, b],$$

(ii) there exists a unique real number $m(a \leq m \leq b)$ such that $\mu(m) = 1$,

(iii) $\mu(t)$ is upper semi-continuous on $[a, b]$,

(iv) $\mu(t)$ is nondecreasing on $[a, m]$ and nonincreasing on $[m, b]$.

The set of all fuzzy numbers is denoted by $\mathbb{F}(\mathbf{R})$. Let ρ denote the Hausdorff distance among bounded closed intervals in$\mathbf{R}$. We introduce a distance on $\mathbb{F}(\mathbf{R})$ by the following:

Definition 2. For two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ in $\mathbb{F}(\mathbf{R})$, the distance $d(\tilde{a}, \tilde{b})$ between $\tilde{a}$ and $\tilde{b}$ is defined by

$$d(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \rho(\tilde{a}_\alpha, \tilde{b}_\alpha),$$

where $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ denote the $\alpha-$cuts of $\tilde{a}$ and $\tilde{b}$, respectively.

Definition 3. For $\varepsilon > 0$ and $\tilde{a} \in \mathbb{F}(\mathbf{R})$, two kinds of $\varepsilon-$neighborhoods of $\tilde{a}$ are defined by

$$\mathrm{B}(\tilde{a}; \varepsilon) = \{\tilde{b} \in \mathbb{F}(\mathbf{R}) \mid d(\tilde{a}, \tilde{b}) < \varepsilon\},$$

$$\overline{\mathrm{B}}(\tilde{a}; \varepsilon) = \{\tilde{b} \in \mathbb{F}(\mathbf{R}) \mid d(\tilde{a}, \tilde{b}) \leq \varepsilon\}.$$

Definition 4. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then

$$\tilde{a} \preceq \tilde{b} \text{ iff } \left(\sup \tilde{a}_\alpha \leq \sup \tilde{b}_\alpha\right) \,\&\, \left(\inf \tilde{a}_\alpha \leq \inf \tilde{b}_\alpha\right) \quad \text{for } \forall \alpha \in [0,1],$$

and

$$\tilde{a} \prec \tilde{b} \text{ iff } \left(\sup \tilde{a}_\alpha < \sup \tilde{b}_\alpha\right) \,\&\, \left(\inf \tilde{a}_\alpha < \inf \tilde{b}_\alpha\right) \quad \text{for } \forall \alpha \in [0,1].$$

Proposition 1. For $\varepsilon > 0$ and $\tilde{a} \in \mathbb{F}(\mathbf{R})$, it holds that

(i) $\tilde{b} \in \overline{\mathrm{B}}(\tilde{a};\varepsilon) \Leftrightarrow \tilde{a}-\varepsilon \preceq \tilde{b} \preceq \tilde{a}+\varepsilon$,

(ii) $\tilde{b} \in \mathrm{B}(\tilde{a};\varepsilon) \Rightarrow \tilde{a}-\varepsilon \prec \tilde{b} \prec \tilde{a}+\varepsilon$.

The condition (iv) in Definition 1 is sometimes exchanged by the following:

(iv)' $\mu(t)$ is strictly increasing on $[a, m]$ and strictly decreasing on $[m, b]$.

Denote the set of all fuzzy sets satisfying (i), (ii), (iii) in Definition 1 and (iv)' by $\mathbb{F}'(\mathbf{R})$. For $\tilde{a} \in \mathbb{F}'(\mathbf{R})$, let

$$\mathrm{B}'(\tilde{a};\varepsilon) = \{\tilde{b} \in \mathbb{F}'(\mathbf{R}) \mid d(\tilde{a}, \tilde{b}) < \varepsilon\}.$$

Proposition 2. For $\varepsilon > 0$ and $\tilde{a} \in \mathbb{F}'(\mathbf{R})$, it holds that

$$\tilde{b} \in \mathrm{B}'(\tilde{a};\varepsilon) \Leftrightarrow \tilde{a}-\varepsilon \prec \tilde{b} \prec \tilde{a}+\varepsilon.$$

Proposition 3. For $\tilde{a} \in \mathbb{F}(\mathbf{R})$, let

$$i(\alpha) = \inf \tilde{a}_\alpha, \quad s(\alpha) = \sup \tilde{a}_\alpha, \quad \alpha \in [0,1].$$

Then $i(\alpha)$ and $s(\alpha)$ are lower semi-continuous and upper semi-continuous on $[0,1]$, respectively.

Proposition 4. Let X be a metric space. Let f_n $(n=1,2,\cdots)$ be a real-valued function defined on X. Suppose that the sequence $\{f_n\}$ converges uniformly to a function f defined on X. If, for each n, f_n is lower (resp. upper) semi-continuous on X, then f is lower (resp. upper) semi-continuous on X.

Theorem 1. ($\mathbb{F}$ $(\mathbf{R})$, d) is a complete metric space.

Definition 5. Let X be a metric space, and let $\tilde{f}$ a mapping from X to $\mathbb{F}(\mathbf{R})$. Let x be a point of X. Then, $\tilde{f}$ is said to be continuous at x, iff for every $\varepsilon > 0$, there exists a positive number $\delta = \delta(x)$ satisfying that

$$y \in S(x;\delta) \Rightarrow \tilde{f}(y) \in B(\tilde{f}(x);\varepsilon).$$

If $\tilde{f}$ is continuous at every x in X, then $\tilde{f}$ is said to be continuous on X.

Proposition 5. Every continuous mapping from a compact metric space X to $\mathbb{F}(\mathbf{R})$ is uniformly continuous on X.

Definition 6. Let X be a metric space. Denote the class of all continuous mappings from X to $\mathbb{F}(\mathbf{R})$ by $C\mathbb{F}[X]$. For two members $\tilde{f}$ and $\tilde{g}$ in $C\mathbb{F}[X]$, define the distance between $\tilde{f}$ and $\tilde{g}$ by

$$\delta(\tilde{f}, \tilde{g}) = \sup_{x \in X} d(\tilde{f}(x), \tilde{g}(x)).$$

Proposition 3. Let X be a compact metric space. Then, for every pair $(\tilde{f}, \tilde{g})$ of fuzzy mappings in $C\mathbb{F}[X]$, $\delta(\tilde{f}, \tilde{g})$ assumes a finite value and is represented by

$$\delta(\tilde{f}, \tilde{g}) = \max_{x \in X} d(\tilde{f}(x), \tilde{g}(x)).$$

Theorem 2. Let X be a compact metric space. Then $(C\mathbb{F}[X], \delta)$ is a complete metric space.

References

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